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Price Postponement in a Newsvendor Model with Wholesale Price-Only Contracts

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Abstract: We consider a variant of the wholesale price-only contract in a simple supply chain consisting of one manufacturer and one retailer, where the manufacturer is the Stackelberg leader and the retailer is the follower. In our model, the manufacturer decides the wholesale price first, and then the retailer chooses his order quantity before the stochastic demand is realized but postpones his pricing decision until after the realization of demand. The existing literature on this model has established structural results under restrictive conditions. In this study, we show that the optimal policies are unique and profit functions are unimodal for both manufacturer and retailer under mild conditions on the demand distribution. We consider both multiplicative and additive demand models. Insights are developed from analyzing the structures of the optimal policies. Our results contribute as well as generalize the existing results in the literature.

Key words: wholesale price-only contract; price-postponement

1. Introduction

Understanding the mechanisms of different contracts is one of the main aspects of supply chain management in today's businesses. Many of these contracts are aiming to coordinate the actions of different partners to achieve higher supply chain performance; the examples are buyback contract, revenue sharing contract, quantity flexibility contract, etc. However, some contracts that do not coordinate the channel are also widely studied in supply chain literature – wholesale price-only contract is one of those. For example, Lariviere and Porteus (2001) point out, "Given the complexity of supply chains, price-only contracts may owe their popularity to their simplicity". Similarly, on page 238, Cachon (2003) states that, "Even though the wholesale-price contract does not coordinate the supply chain, the wholesale-price contract is worth studying because it is commonly observed in practice".

In this paper, we consider a variant of the wholesale price-only contract in a simple supply chain consisting of one manufacturer and one retailer, wherein the manufacturer is the Stackelberg leader and the retailer is the follower. In our model, the manufacturer decides her wholesale price first, and then the retailer chooses his order quantity before the stochastic demand is realized but postpones his pricing decision until after the realization of demand. In the literature (e.g., Van Mieghem and Dada 1999, Granot and Yin 2008), this is coined the *price-postponement* model.

To mitigate the effect of demand uncertainty and be flexible enough to offer a broad range of products, postponement strategies have recently become a popular business concept that are used in many supply chains. There are four types of postponement strategies, namely, pull postponement, logistics postponement, form postponement and price postponement (see Cheng et al. (2010) for the details on each of these strategies). The first three strategies are also referred to as the production postponement (Van Mieghem and Dada 1999). Based on the observations of Van Mieghem and Dada, one advantage of the price postponement strategy over the production postponement strategy is that it makes capacity investment and production (inventory) decisions relatively insensitive to demand uncertainty, since profit margin can be covered by suitably charging the selling price after demand is realized. Thus, price postponement can help to reduce market risks while taking other strategic and tactical decisions before the realization of stochastic demand. In some manufacturing contexts, marginal costs of production may increase under production postponement due to the requirement of faster response time once production starts. In such a case, price postponement may be preferred over the production postponement. Another advantage of price postponement is its ease of implementation. Unlike the production postponement strategy, that requires re-engineering techniques such as operations reversal and standardization of product and process, price postponement is a managerial decision that is determined by marketing managers.

Realistic variations of price postponement strategies are used by GreatModels.com, an online retail store selling many items such as books, accessories, decals, airbrush, compressors, and scale models of aircrafts, helicopters, missiles, ships, cars, etc. (Granot and Yin 2008). This retailer determines or readjusts the price of an item after demand information from pre-orders is observed. As an example of price postponement strategy, Van Mieghem and Dada (1999) mentions the case of an automobile dealership who must decide on the number of cars to stock before market demand is known. Then, the selling price can be negotiated with the customers during the sales. Another realistic variation of price postponement strategy was used by Bank of China (BOC) Hong Kong in July 2002. In the face of high demand uncertainty, BOC set an initial public offering price range, between HK$6.93 and HK$9.5 per share, for investors to subscribe to its shares. Since the public offering was over-subscribed by 26 times, BOC finally allotted at least 500 shares to each investor at a price of HK$8.5 (Cheng et al. 2010). It is quite possible that similar strategy of offering a price range initially and then, after observing the market demand, adjusting the "final" price later, could very well be used for expensive products such as aircrafts, ships, missiles, cars, etc. These examples show the potential applications of the price postponement strategy. As the postponement strategies are becoming more and more popular day by day with the advancement of information technology, and since the price postponement strategy has multiple benefits as mentioned above, it is worthwhile to study this strategy in some detail. Moreover, wholesale price-only contract is the one which is most popular and commonly used in practice (Lariviere and Porteus 2001, Cachon 2003), yet theoretical results are far from complete for this case as can be seen from the limited results available in the literature so far (see, e.g., Granot and Yin 2008). Therefore, it is important for researchers to investigate the combined model of price postponement strategy under the wholesale price-only contract.

In our study of the price-postponement model, for the multiplicative demand case, we establish the unimodality of the expected profit functions of both manufacturer and retailer, and derive the unique optimal wholesale price (w) and retail order quantity (Q) under mild conditions on the demand distribution. Specifically, for the linear demand function, the condition on the probability distribution is either IGFR (as defined in Lariviere and Porteus 2001) or a generalized version of a condition studied in Ziya et al. (2004); for the exponential demand function, the condition is very general and includes all distributions for which generalized failure rate is monotone; and for the isoelastic demand function, no condition is needed on the distribution. The results on the first two cases provide a significant generalization over Granot and Yin (2008, see Table 3) who, for the analogous models with wholesale price-only contracts, established unimodality of the manufacturer's expected profit function under the restriction that $\varepsilon f(\varepsilon)$ is increasing in ε, where $f(\varepsilon)$ represents the probability density function of $\varepsilon \in [A, B]$ with $A \geq 0$ and $B < \infty$.

Apparently, one distribution that satisfies the above condition of Proposition 2 in Granot and Yin (2008) is the beta distribution with restricted values (≤ 1) for one of its two parameters that include the power and uniform distributions. These distributions have bounded support. Note that other commonly used distributions, such as gamma, Weibull, normal, lognormal, Pareto, etc., do not usually satisfy the condition "$\varepsilon f(\varepsilon)$ is increasing in ε", unless, as the authors mentioned, the domain of ε is relatively small. However, the market demand or the optimal stocking level may not always correspond to the values of ε in this small domain. In this paper, for both linear and exponential demand functions with multiplicative model, we are able to generalize Granot and Yin (2008)'s result on the unimodality of the manufacturer's expected profit function for classes of distributions that include all IGFR distributions which, as noted by Lariviere and Porteus (2001), include most of the common distributions. We also provide the example of a DGFR distribution (defined in Lariviere 2006) for which our unimodality results will apply. The condition for the case of exponential demand function is even weaker than requiring that the generalized failure rate is monotone; hence all IGFR and DGFR distributions will satisfy this condition. Such generalizations make the price postponement model more viable to use, since under unimodality, both the optimal *w* and *Q* are unique and can be expressed in simple analytical forms that are easily computable, as can be seen from our propositions.

Similar to the multiplicative demand model, the case of additive model also continues to interest the researchers. However, no result is available for this model under the wholesale price-only contract and is recognized to be a difficult case in the literature (Wang et al. 2004, Granot and Yin 2008 and Song et al. 2008). For the linear demand function, here we prove the unimodality of the manufacturer's and retailer's expected profit functions and derive the unique optimal wholesale price and retail order quantity for IFR demand distributions. We also provide counter-examples to show how the unimodality of the manufacturer's expected profit function may fail for some DFR distributions that are IGFR. Note that we do not consider the cases of exponential and isoelastic functions because they are not meaningful when the demand model is additive (Petruzzi and Dada 2002).

Finally, we compare the optimal policies for the four models we study here to gain insights into the structures of the solutions and their implications. In particular, we explore why, to establish the unimodality of the manufacturer's expected profit function, different sufficient conditions on the distributions are needed for different demand models.

The remaining part of the paper is organized as follows. In Section 2, we describe the model. The analysis of both multiplicative and additive demand models and insights into their solutions are provided in Section 3. Proofs of all the results are included in the Appendix.

2. Model Description

We consider the multiplicative and additive demand models for the form $X = D(p) \cdot u$ and $X = D(p) + u$ respectively, where $D(p)$ is the deterministic part of demand *X* that decreases in the retail price *p*, and *u* is the random part of *X* with a support on (A, B), where $A \geq 0, B \leq \infty$.

The events, contract and cost parameters in our model occur in the following sequence. A manufacturer with unlimited capacity produces items at a cost of *c* per unit and acting as a Stackelberg leader offers to charge a per unit wholesale price *w* from a retailer. The retailer then decides his optimal order quantity *Q* for the selling season. Subsequently, the demand is realized. Based on the demand realization, retailer chooses his retail price *p* that maximizes his profit. For simplicity, we assume that the salvage value of any unsold inventory is zero and that any unsatisfied demand is lost without any shortage penalty. To avoid trivial solutions, we let $c < w < p$. Note that, the analysis of our results would not change if we allow for positive salvage value and shortage penalty.

For the convenience of analysis, let $F(u)$ and $f(u)$ respectively denote the distribution and density functions of u, the random component of the demand. To avoid technicalities, unless stated otherwise, we will assume the following:

Assumption 1. *The probability density function* $f(u)$ *exists and is differentiable.*

Now, letting $\bar{F}(u) = 1 - F(u)$, we denote the *failure rate* of u as $h(\xi) = \dfrac{f(\xi)}{\bar{F}(\xi)}$ and the *generalized failure rate* of u as $g(\xi) = \xi h(\xi)$. To clearly specify the probability distributions to be used in our analysis, let us now define the following classes of distributions:

Definition 1. (Ross 1996) *The random variable u is said to have an increasing failure rate (IFR) distribution if* $h(\xi)$ *is non-decreasing in* (A, B) *and a decreasing failure rate (DFR) distribution if* $h(\xi)$ *is non-increasing in* (A, B).

Definition 2. (Lariviere and Porteus 2001, Lariviere 2006) *The random variable u is said to have an increasing generalized failure rate* (*IGFR*) *distribution if* $g(\xi)$ *is non-decreasing in* (A, B) *and a decreasing generalized failure rate* (*DGFR*) *distribution if* $g(\xi)$ *is non-increasing in* (A, B).

Most common distributions are IGFR (Lariviere and Porteus 2001). IFR is a large subset of IGFR and includes normal, truncated normal, uniform, gamma with shape parameter ≥ 1, Weibull with shape parameter ≥ 1, etc. (Porteus 2002). Examples of DFR distributions are gamma and Weibull with shape parameter ≤ 1. While DGFR distributions may be rare, later we will show an example of a DGFR distribution.

Finally, we write the profit functions of the channel members. In our decentralized system with the wholesale price-only contract, the retailer's expected profit function is given by

$$\Pi_R(p, Q) = E[p \min(Q, X)] - wQ,$$

and the manufacturer's expected profit function is given by

$$\Pi_M(w, Q) = (w - c)Q.$$

The results and insights of our analysis are provided next.

3. The Analysis, Results and Insights

In this section, we derive our results for the price-postponement model with wholesale price-only contracts. The linear, exponential and isoelastic demand functions for the multiplicative model and the linear demand function for the additive model are discussed in the following three subsections.

3.1 The Multiplicative Demand Model

For the multiplicative demand model $X = D(p) \cdot u$, we show that when the demand function is linear, that is, $D(p) = 1 - p$, the manufacturer's profit function is unimodal in the retailer's order quantity for IGFR demand distributions as well as for demand distributions satisfying another very general condition which holds for most of the common distributions. Next, we establish unimodality of the profit functions and derive unique optimal solutions respectively for the exponential demand function under a very mild condition on the generalized failure rate of *u* and for the isoelastic demand function under no condition on the distribution of *u*. To proceed, we start with the linear demand function.

The Case of Linear Demand Function

When $D(p) = 1 - p$, by Lemma 1 of Granot and Yin (2008), we know that the optimal retail price is given by

$$p^* = \begin{cases} 1/2 & \text{if } \hat{u} \leq 2Q \\ 1 - Q/\hat{u} & \text{if } \hat{u} \geq 2Q, \end{cases} \tag{1}$$

where $\hat{u}$ is the realized value of *u*, the random part of the demand. Also, from (13) and (14) of Granot and Yin (2008), the retailer's optimal order quantity *Q* is given by

$$w = \bar{F}(2Q) - 2Q\int_{2Q}^{B} \frac{f(u)}{u} du \,,$$

and the manufacturer's expected profit function is given by

$$\Pi_M(Q) = (w - c)Q = \left[\bar{F}(2Q) - 2Q\int_{2Q}^{B} \frac{f(u)}{u} du - c \right] Q. \tag{2}$$

While the concavity of the retailer's expected profit function has been established in Lemma 2 of Granot and Yin (2008) for general demand distributions, the unimodality of the manufacturer's expected profit function $\Pi_M(Q)$ and the existence of unique optimal solutions have only been established for demands for which *uf*(*u*) is increasing in $u \in [A, B]$ (see their Proposition 2). We significantly generalize this later result in the following proposition. By suitably analyzing the behaviors of the first, second and third derivatives of $\Pi_M(Q)$ and their limits as $Q \to 0^+$ and $Q \to +\infty$, we show that

Proposition 1. *Under Assumption 1, the manufacturer's expected profit function* $\Pi_M(Q)$ *is unimodal in Q if the distribution of u is either*

(i) IGFR , or

(ii) $\dfrac{2Qf'(2Q)}{f(2Q)} + 2$*, as a function of Q, changes sign at most once.*

Therefore, the optimal order quantity Q^* *of the retailer and the optimal wholesale price* w^* *of the manufacturer are unique and given by*

$$c = \bar{F}(2Q^*) - 4Q^* \int_{2Q^*}^{B} \frac{f(u)}{u} du \quad \text{and} \quad w^* = \frac{1}{2}\left[c + \bar{F}(2Q^*)\right]. \tag{3}$$

The first condition (IGFR) is the same as that of Condition C3 in Ziya et al. (2004, Proposition 4.1). The second condition is a generalization of Condition C2 in Ziya et al., where their C2 is equivalent to our case 2a) that $\dfrac{2Qf'(2Q)}{f(2Q)} + 2 \geq 0$ (which is a sub-case of our condition (ii), see the proof of Proposition 1 in the Appendix). To see how general the condition (ii) is, consider the counter-example given in Section 6 of Ziya et al. (2004) with the probability distribution $F(x) = 1 - (x - v)^{-2}$, $x \in [1 + v, +\infty)$, $0 < v < \infty$, which satisfies neither their Condition C2 nor C3 (IGFR); however, this distribution satisfies our case 2b) that $\dfrac{2Qf'(2Q)}{f(2Q)} + 2 \leq 0$ and moreover it's a DGFR distribution. For this distribution, by our Proposition 1, the manufacturer's expected profit function is unimodal in *Q*.

Further note that, as also mentioned in Lariviere and Porteus (2001), Condition C2 can fail for many common distributions, such as exponential, normal and Gamma distributions for which Condition C3 as well as our condition (ii) hold. For example, with the exponential density $f(x) = \lambda e^{-\lambda x}$, we get $\frac{xf'(x)}{f(x)} + 2 = 2 - \lambda x$, which is not nonnegative for all *x*. However, it satisfies condition (ii) that $\frac{xf'(x)}{f(x)} + 2$ changes sign at most once. This shows our condition (ii) is a significant generalization of Condition C2.

Ziya et al. (2004) have pointed out that neither of their Condition C2 nor C3 is more restrictive than the other. The same is also true with our conditions (i) and (ii) in Proposition 1. While we could not prove whether condition (ii) is more general than the IGFR condition (i), it is worth noting that our condition (ii) is satisfied by the long list of distributions provided in Bagnoli and Bergstrom (2005), which includes the commonly used log-concave and log-convex densities. If we let S_1 be the class of probability distributions that satisfies either condition (i) or condition (ii) of Proposition 1, then it is evident that, with our previous example of the DGFR distribution, the class S_1 is strictly bigger than the class of all IGFR distributions. Thus, Proposition 1 establishes the unimodality of the manufacturer's expected profit function and provides the unique optimal solutions for a wide class of distributions.

The Case of Exponential Demand Function

When $D(p) = e^{-p}$, the optimal retail price is given by (a derivation is shown in the Appendix)

$$p^* = \begin{cases} 1 & \text{if } \hat{u} \le eQ \\ \ln\left(\hat{u}/Q\right) & \text{if } \hat{u} \ge eQ, \end{cases} \tag{4}$$

where $\hat{u}$ is the realization of *u* and "ln" represents the natural logarithm.

Then, the expected profit of the retailer is given by

$$\tilde{\Pi}_R(Q) = E[p\min(Q,X)] - wQ$$
$$= \int_A^{eQ}[1\cdot e^{-1}\cdot u]f(u)du + \int_{eQ}^{B}[(\ln u - \ln Q)\cdot \frac{Q}{u}\cdot u]f(u)du - wQ \quad (5)$$
$$= e^{-1}\int_A^{eQ} uf(u)du + Q\int_{eQ}^{B}\ln u\cdot f(u)du - Q\ln Q\cdot \bar{F}(eQ) - wQ.$$

Subsequently, from (5), the first order condition $\frac{d\tilde{\Pi}_R}{dQ} = 0$ gives the optimal wholesale price w as

$$w = \int_{eQ}^{B}\ln u\cdot f(u)du - [\ln Q + 1]\bar{F}(eQ). \quad (6)$$

Therefore, the expected profit function of the manufacturer is given by

$$\tilde{\Pi}_M(Q) = (w-c)Q = \left[\int_{eQ}^{B}\ln u\cdot f(u)du - \ln eQ\cdot \bar{F}(eQ) - c\right]Q. \quad (7)$$

Now, by analyzing the first and second derivatives of $\tilde{\Pi}_M(Q)$ and their limits as $Q \to 0^+$ and $Q \to +\infty$, we prove that

Proposition 2. *Under Assumption 1, (i) The retailer's expected profit function is strictly concave in Q, and (ii) The manufacturer's expected profit function* $\tilde{\Pi}_M(Q)$ *is unimodal in Q if the distribution of u satisfies the following condition C:*

(C) $\frac{eQf(eQ)}{\bar{F}(eQ)} - 1$, *as a function of Q, changes sign at most once.*

Therefore, the optimal order quantity Q^* *of the retailer and the optimal wholesale price* w^* *of the manufacturer are unique and given by*

$$c = \int_{eQ^*}^{B}\ln u. f(u)du - [2 + \ln Q^*]\bar{F}(eQ^*) \quad \text{and} \quad w^* = c + \bar{F}(eQ^*). \quad (8)$$

Once again, the above result overcomes the restriction of "$uf(u)$ is increasing in u" in Granot and Yin (2008, see Table 3). Condition C is weaker than requiring that $\frac{eQf(eQ)}{\bar{F}(eQ)}$ is monotone. Let S_2 be the class of probability distributions satisfying condition C. Clearly, S_2 includes all IGFR and DGFR distributions. Thus, by Proposition 2, for the distribution $F(x) = 1-(x-v)^{-2}$, $x \in [1+v, +\infty)$, $0 < v < \infty$, which is DGFR, the manufacturer's expected profit function is unimodal in Q and the optimal solutions are uniquely obtained from (8).

The Case of Isoelastic Demand Function

For the isoelastic demand function $D(p) = p^{-q}$, $q > 1$, the optimal retail price is given by

$$p^* = \left(\hat{u} / Q\right)^{1/q}, \tag{9}$$

where $\hat{u}$ is the realized value of *u*. The following general result is shown in Granot and Yin (2008). A proof is included in the Appendix for facilitating some of our explanations in subsection 3.3.

Proposition 3. *For any distribution of u, the expected profit functions of both the manufacturer and retailer are strictly concave in the retailer's order quantity Q. Therefore, the optimal order quantity* Q^* *and the optimal wholesale price* w^* *are unique and given by*

$$c = \frac{(q-1)^2}{q^2}\left(Q^*\right)^{-1/q}\int_A^B u^{1/q}\cdot f(u)\,du \quad \text{and} \quad w^* = \frac{qc}{q-1}. \tag{10}$$

From the proof we can observe that Proposition 3 holds even for discrete distributions, but with a continuous decision variable *Q*. The underlying reason will be explained in subsection 3.3. Now that we have analyzed the multiplicative demand model, we discuss the case of additive demand model next.

3.2 The Additive Demand Model

We now analyze the additive demand model $X = D(p) + u$. While an interest in the additive demand case is always shown in the literature (e.g., Wang et al. 2004, Granot and Yin 2008, Song et al. 2008), no detail discussion or preliminary results are available for this model under the wholesale price-only contract. Here, for the linear demand function, with IFR demand distributions, we show that the retailer's expected profit function is strictly concave and the manufacturer's expected profit function is unimodal in the retail order quantity. Subsequently, we derive the unique optimal wholesale price of the manufacturer, and the optimal order quantity and price of the retailer. Note that, as discussed in Petruzzi and Dada (2002), if one considers the exponential and isoelastic demand functions for the additive model, then demand $X > 0$ even when $p \to \infty$ if the realized $u > 0$, so that infinite profit is possible. Therefore, we do not analyze these cases since they are not natural models for demand.

The Case of Linear Demand Function

When $D(p) = 1 - p$, the optimal retail price is given by (the derivation is provided in the Appendix)

$$p^* = \begin{cases} \dfrac{1+\hat{u}}{2} & \text{if } \hat{u} \le 2Q-1 \\ 1+\hat{u}-Q & \text{if } \hat{u} \ge 2Q-1. \end{cases} \tag{11}$$

where $\hat{u}$ is the realized value of u.

Then, the expected profit of the retailer is given by

$$\Pi_R(Q) = \int_A^{2Q-1} \left[\frac{1+u}{2} \cdot \frac{1+u}{2} \right] f(u)du + \int_{2Q-1}^{B} (1+u-Q) \cdot Q \cdot f(u)du - wQ. \tag{12}$$

From (12), solving the first order condition $\dfrac{d\Pi_R}{dQ} = 0$, we get the optimal wholesale price w as

$$w = \int_{2Q-1}^{B} (1+u) f(u)du - 2Q \cdot \bar{F}(2Q-1). \tag{13}$$

Thus, the manufacturer's expected profit function is given by

$$\Pi_M(Q) = (w-c)Q = \left[\int_{2Q-1}^{B} (1+u) f(u)du - 2Q \cdot \bar{F}(2Q-1) - c \right] Q, \tag{14}$$

so that

$$\frac{d\Pi_M}{dQ} = \int_{2Q-1}^{B} (1+u) f(u)du - 4Q\bar{F}(2Q-1) - c, \tag{15}$$

and

$$\begin{aligned} \frac{d^2\Pi_M}{dQ^2} &= 4Qf(2Q-1) - 4\bar{F}(2Q-1) \\ &= 2\bar{F}(2Q-1)\left[\frac{2Qf(2Q-1)}{\bar{F}(2Q-1)} - 2 \right] \\ &= 2\bar{F}(2Q-1)\left[\frac{(2Q-1)f(2Q-1)}{\bar{F}(2Q-1)} + \frac{f(2Q-1)}{\bar{F}(2Q-1)} - 2 \right]. \end{aligned} \tag{16}$$

With these, we establish the following result:

Proposition 4. *Under Assumption 1, (i) The retailer's expected profit function is strictly concave in Q, and (ii) The manufacturer's expected profit function* $\Pi_M(Q)$ *is unimodal in Q if the distribution of u is IFR. Therefore, the optimal order quantity* Q^* *of the retailer and the optimal wholesale price* w^* *of the manufacturer are unique and given by*

$$c = \int_{2Q^*-1}^{B} (1+u) f(u) du - 4Q^* \bar{F}(2Q^*-1) \quad \text{and} \quad w^* = c + 2Q^* \bar{F}(2Q^*-1) \,. \tag{17}$$

Note that, unlike Proposition 1, we cannot relax the assumption of IFR distribution on *u* to IGFR distribution. The reason is that IGFR distributions also include DFR distributions (Lariviere 2006); and if that is the case, then in $\left[\frac{(2Q-1)f(2Q-1)}{\bar{F}(2Q-1)} + \frac{f(2Q-1)}{\bar{F}(2Q-1)} - 2\right]$ term of (16) we have, $\frac{(2Q-1)f(2Q-1)}{\bar{F}(2Q-1)}$ is non-decreasing in *Q*, while $\frac{f(2Q-1)}{\bar{F}(2Q-1)}$ is non-increasing in *Q*, so that the monotonicity of $\left[\frac{(2Q-1)f(2Q-1)}{\bar{F}(2Q-1)} + \frac{f(2Q-1)}{\bar{F}(2Q-1)} - 2\right]$ and hence, $\frac{d^2\Pi_M}{dQ^2}$ is not guaranteed. Subsequently, $\frac{d\Pi_M}{dQ}$ may have multiple zeroes. We actually observe these behaviors for some gamma and Weibull distributions with shape parameter < 1, for which these distributions are both DFR as well as IGFR. For the gamma distribution with shape parameter 2/3 and scale parameter 0.6, with the manufacturer's production cost *c* = 0.12, we plot the graphs of $\frac{d\Pi_M}{dQ}$ and $\frac{d^2\Pi_M}{dQ^2}$ in Figures 1a-1b and 2a-2b, respectively (shown in the Appendix). From Figures 1a and 1b, we observe that, in the interval [0.495, 0.540] for *Q*, $\frac{d\Pi_M}{dQ}$ has three zeroes, namely, 0.4978, 0.5011 and 0.5320. Moreover, from Figures 2a and 2b, we observe that, in the above interval, $\frac{d^2\Pi_M}{dQ^2}$ is not monotone. Very similar behaviors are also observed for the Weibull distribution with shape parameter 0.8 and scale parameter 1, and with the manufacturer's production cost c = 0.138. For this case, in the interval [0.498, 0.508] for Q, $\frac{d\Pi_M}{dQ}$ has three zeroes at 0.4987, 0.5038 and 0.5072. A detailed explanation for these behaviors of multiple zeroes for $\frac{d\Pi_M}{dQ}$ and non-monotonicity of $\frac{d^2\Pi_M}{dQ^2}$ is provided in the Appendix. These examples show that Proposition 4 does not hold for some DFR distributions that are also IGFR. Therefore, we preclude establishing unimodality of $\Pi_M(Q)$ for all IGFR distributions and restrict our results in Proposition 4 to IFR distributions.

3.3 Insights and Conclusions

We now compare the results for the above four models to garner some insights. Let us first consider the cases of linear and exponential demand functions for the multiplicative model. From (1) and (4) we observe that the optimal retail price in both cases has similar structure, namely, for relatively high realizations of the random component of the demand (i.e., $\hat{u} \geq 2Q$ and $\hat{u} \geq eQ$, respectively for linear and exponential demand functions), the retailer charges the price that clears the market so that there is no unsatisfied demand. On the other hand, for relatively low realizations of u (i.e., $\hat{u} \leq 2Q$ and $\hat{u} \leq eQ$, respectively), the retailer charges the price at which his profit achieves the maximum value. For both demand functions, it's a constant price (1/2 and 1, respectively) and the retailer may have some unsold stock. While the manufacturer's optimal wholesale prices also have similar structure for both these demand functions (see (3) and (8)), the structures of the retailer's optimal order quantities are slightly different and it is primarily due to the specific form of the corresponding demand function. For the isoelastic demand function with multiplicative model, the structures of the solutions are quite different. The optimal retail price in this case has a unique form for all values of the realization $\hat{u}$ (see (9)). This allows the manufacturer's wholesale price to be initially written in terms of a moment of the distribution (see (32)) which, consequently, after substituting out the expression for the optimal order quantity, leads to a distribution-free expression for the optimal wholesale price. Moreover, notice from (10) that the retailer's optimal order quantity is written in terms of a moment of the distribution. These special forms of the optimal solutions lead to the unimodality of the manufacturer's profit function for all distributions, both continuous as well as discrete. On the other hand, for the linear and exponential demand functions, the optimal retail prices change from a constant value to an increasing function of $\hat{u}$, once $\hat{u}$ exceeds $2Q$ and eQ, respectively. This makes the corresponding wholesale prices to depend on the tail probabilities $\bar{F}(2Q)$ and $\bar{F}(eQ)$, respectively (see (3) and (8)), which, subsequently, prompt us to find the regularity conditions that will lead to the unimodality of the manufacturer's expected profit function. Such exploration results in sufficient conditions based on the generalized failure rate of the distributions, as we observe in Propositions 1 and 2 (except condition (ii) of Proposition 1 that come from studying the regularity properties of the second and third derivatives of the manufacturer's expected profit function).

Next, for the additive model with linear demand function, from (11) we observe that the optimal retail price for relatively high realizations of *u* (i.e., $\hat{u} \geq 2Q-1$) is the price that clears the market. And for relatively low realizations of *u* (i.e., $\hat{u} \leq 2Q-1$), the retailer charges the price that maximizes his profit. Notice that this price is $\frac{1+\hat{u}}{2}$ which depends on the realized value of *u* and is not a constant as was the case for the multiplicative model with linear demand function. This sensitivity of the optimal retail price for lower values of the realization of *u* influences the wholesale price (see (12) and (13)) in such a way that it can lead to non-monotonicity for the second derivative and multiple optima for the manufacturer's expected profit function if the distribution is DFR, as illustrated with two examples in the previous subsection. From the explanation provided in the Appendix, we observe that one of the reasons for the above non-monotonicity and multi-modality is the sharp change in the density function for smaller values of *u* when its distribution is DFR (see Johnson et al. 2004, p. 341 and 631, respectively for the gamma and Weibull densities with shape parameter < 1). For this reason, to establish the unimodality of the manufacturer's expected profit function, we had to restrict to IFR distributions.

Finally, let us compare the optimal retail prices between the multiplicative and additive demand models. When the realizations of u are relatively high so that the retailer charges the market clearing prices, the optimal retail prices for all three multiplicative demand models are expressed as a function of the ratio between $\hat{u}$ and Q, whereas for the additive model it is expressed as a function of the difference between $\hat{u}$ and Q. Clearly, these forms come directly from the multiplicative or additive structure of the model itself. However, when the realizations of u are relatively low in which case the retailer charges the price that maximizes his profit, the characteristics of the optimal retail prices are quite different between the multiplicative and additive demand models. For the multiplicative model with linear demand function, the optimal retail price is 1/2, a fixed number. In contrast, for the additive model, it is $\frac{1+\hat{u}}{2}$ which can take any value between (1/2, Q) since $\hat{u}$ can vary between 0 and $2Q-1$ in this case. This shows that not only the values of the optimal retail prices differ between the multiplicative and additive demand models, but also their structures are different. Consequently, this creates some structural differences in the optimal solutions Q^* and w^* between these two models (see (3) and (17)) as a result of which we get different sufficient conditions on the distributions to guarantee the unimodality of the manufacturer's expected profit function.

To conclude, here we have studied the price-postponement model for a newsvendor problem with wholesale price-only contract. For both multiplicative and additive demand models, we have shown that the optimal policies and expected profit functions of the manufacturer and retailer are well-behaved under reasonably mild conditions on the demand distribution. Extension of this research to the *no-postponement* model in which the optimal retail price is also decided before the realization of demand eludes us at the moment. While a complete set of results for this model under buyback contracts is provided in Song et al. (2008) and Granot and Yin (2008) for the multiplicative demand case, the corresponding results under the wholesale price-only contract do not follow from there and most probably would require a different technique to solve the problem.

Appendix

Proof of Proposition 1.

In (2), we have the manufacturer's expected profit function as

$$\Pi_M(Q) = \left[\bar{F}(2Q) - 2Q\int_{2Q}^{B} \frac{f(u)}{u} du - c \right] Q ,$$

so that the first derivative of Π_M is given by

$$\frac{d\Pi_M}{dQ} = \bar{F}(2Q) - 4Q\int_{2Q}^{B} \frac{f(u)}{u} du - c. \tag{18}$$

In (18), we can write

$$\begin{aligned}
-4Q\int_{2Q}^{B} \frac{f(u)}{u} du = 4Q\int_{2Q}^{B} \frac{1}{u} d\bar{F}(u) &= 4Q\left[\frac{\bar{F}(u)}{u} \Big|_{2Q}^{B} - \int_{2Q}^{B} \bar{F}(u) d\,\frac{1}{u} \right] \\
&= 4Q\left[-\frac{\bar{F}(2Q)}{2Q} + \int_{2Q}^{B} \frac{\bar{F}(u)}{u^2} du \right] \\
&= -2\bar{F}(2Q) + 4Q\int_{2Q}^{B} \frac{\bar{F}(u)}{u^2} du,
\end{aligned}$$

which implies that

$$\bar{F}(2Q) = 2Q\int_{2Q}^{B} \frac{\bar{F}(u)}{u^2} du + 2Q\int_{2Q}^{B} \frac{f(u)}{u} du .$$

Hence, (18) can be written as

$$\begin{aligned}
\frac{d\Pi_M}{dQ} &= \bar{F}(2Q) - 4Q\int_{2Q}^{B} \frac{f(u)}{u} du - c \\
&= 2Q\int_{2Q}^{B} \frac{\bar{F}(u)}{u^2} du + 2Q\int_{2Q}^{B} \frac{f(u)}{u} du - 4Q\int_{2Q}^{B} \frac{f(u)}{u} du - c \\
&= 2Q\left[\int_{2Q}^{B} \frac{\bar{F}(u)}{u^2} du - \int_{2Q}^{B} \frac{f(u)}{u} du - \frac{c}{2Q} \right].
\end{aligned}$$

If we define: $L(Q) = \int_{2Q}^{B} \frac{\bar{F}(u)}{u^2} du - \int_{2Q}^{B} \frac{f(u)}{u} du - \frac{c}{2Q}$,

then $\frac{d\Pi_M}{dQ} = 2Q \cdot L(Q)$, and the first derivative of $L(Q)$ equals

$$\frac{dL(Q)}{dQ} = -\frac{\bar{F}(2Q)}{2Q^2} + \frac{f(2Q)}{Q} + \frac{c}{2Q^2} = \frac{\bar{F}(2Q)}{2Q^2}\left[\frac{2Q \cdot f(2Q)}{\bar{F}(2Q)} + \frac{c}{\bar{F}(2Q)} - 1 \right]. \tag{19}$$

Notice that in (19), $\frac{c}{\bar{F}(2Q)}$ is non-decreasing in *Q*; and if the demand distribution is IGFR, then $\frac{2Q\cdot f(2Q)}{\bar{F}(2Q)}$ is also non-decreasing in *Q*. Moreover,

$$\lim_{Q\to 0^+}\left[\frac{2Q\cdot f(2Q)}{\bar{F}(2Q)}+\frac{c}{\bar{F}(2Q)}-1\right]=0+c-1<p-1\leq 0,$$

$$\lim_{Q\to +\infty}\left[\frac{2Q\cdot f(2Q)}{\bar{F}(2Q)}+\frac{c}{\bar{F}(2Q)}-1\right]=+\infty-1>0,$$

which means that $\frac{dL(Q)}{dQ}$ changes sign only once, from negative to positive.

Now, note that

$$\frac{d\Pi_M}{dQ}|_{Q\to 0^+}=\lim_{Q\to 0^+} 2Q\cdot L(Q)=1-c>0, \tag{20}$$

and

$$\frac{d\Pi_M}{dQ}|_{Q\to +\infty}=\lim_{Q\to +\infty} 2Q\cdot L(Q)\leq -c<0, \tag{21}$$

which implies that $L(0)>0$ and $L(+\infty)<0$. Since $\frac{dL(Q)}{dQ}$ changes sign only once from negative to positive, $L(Q)$ changes sign exactly once from positive to negative, and so is $\frac{d\Pi_M}{dQ}$. Therefore, we can conclude that if the demand distribution is IGFR, then $\Pi_M(Q)$ is unimodal and there exists a unique *Q* which maximizes $\Pi_M(Q)$.

Next, we will show that under condition (ii), $\Pi_M(Q)$ is again unimodal.

For this purpose, we need to analyze the second derivative of Π_M:

$$\begin{aligned}\frac{d^2\Pi_M}{dQ^2}&=2f(2Q)-4\int_{2Q}^{B}\frac{f(u)}{u}du\\&=2\int_{2Q}^{B}\frac{f(u)}{u}du\left\{\frac{f(2Q)}{\int_{2Q}^{B}\frac{f(u)}{u}du}-2\right\}.\end{aligned} \tag{22}$$

We will analyze the sign of $\frac{d^2\Pi_M}{dQ^2}|_{Q=0^+}$ separately in three cases below:

1a) If $f(0)=0$, then

$$\lim_{Q\to 0^+}\frac{d^2\Pi_M}{dQ^2}=0-\lim_{Q\to 0^+} 4\int_{2Q}^{B}\frac{f(u)}{u}du<0.$$

1b) If $f(0)>0$ and $f(0)$ is finite, then

$$\lim_{Q\to 0^+}\frac{d^2\Pi_M}{dQ^2}$$

$$=\lim_{Q\to 0^+} 2f(2Q)-\lim_{Q\to 0^+} 4\int_{2Q}^{B}\frac{f(u)}{u}du$$

$$=\lim_{Q\to 0^+} 2f(2Q)-\lim_{Q\to 0^+} 4\int_{2Q}^{1}\frac{f(u)}{u}du-4\int_{1}^{B}\frac{f(u)}{u}du$$

$$=\lim_{Q\to 0^+} 2f(2Q)-4f(\tilde{u})\times\lim_{Q\to 0^+}\int_{2Q}^{1}\frac{1}{u}du-4\int_{1}^{B}\frac{f(u)}{u}du \quad \text{(by mean value theorem of integrals)}$$

$$=\lim_{Q\to 0^+} 2f(2Q)-4f(\tilde{u})\times\lim_{Q\to 0^+}\ln u\,\Big|_{2Q}^{1}-4\int_{1}^{B}\frac{f(u)}{u}du$$

$$=\lim_{Q\to 0^+} 2f(2Q)-(+\infty)-4\int_{1}^{B}\frac{f(u)}{u}du$$

$$=-\infty, \tag{23}$$

where $0<\tilde{u}<1$. From (23) we also have: $\lim_{Q\to 0^+} 4\int_{2Q}^{B}\frac{f(u)}{u}du=+\infty$.

1c) If $\lim_{Q\to 0^+} f(2Q)=+\infty$, then using L'Hospital rule, from (22) we get

$$\begin{aligned}\lim_{Q\to 0^+}\frac{d^2\Pi_M}{dQ^2}&=\lim_{Q\to 0^+}\left[2\int_{2Q}^{B}\frac{f(u)}{u}du\right]\cdot\left\{\lim_{Q\to 0^+}\frac{f(2Q)}{\int_{2Q}^{B}\frac{f(u)}{u}du}-2\right\}\\&=\lim_{Q\to 0^+}\left[2\int_{2Q}^{B}\frac{f(u)}{u}du\right]\cdot\left\{\lim_{Q\to 0^+}\frac{\frac{df(2Q)}{dQ}}{\frac{d}{dQ}\int_{2Q}^{B}\frac{f(u)}{u}du}-2\right\}\\&=\lim_{Q\to 0^+}\left[2\int_{2Q}^{B}\frac{f(u)}{u}du\right]\cdot\left\{-\lim_{Q\to 0^+}\frac{2Q\cdot f'(2Q)}{f(2Q)}-2\right\}.\end{aligned} \tag{24}$$

Now, we are ready to analyze $\lim_{Q\to+\infty}\frac{d^2\Pi_M}{dQ^2}$. From the theory of probability, we know that

$\lim_{Q\to+\infty} f(2Q)=0$ for any distribution. Besides,

$\lim_{Q\to+\infty} 4\int_{2Q}^{B}\frac{f(u)}{u}du<\lim_{Q\to+\infty} 4\int_{2Q}^{B} f(u)du=\lim_{Q\to+\infty} 4\bar{F}(2Q)=0$. Hence, from (22) we get

$$\lim_{Q\to+\infty}\frac{d^2\Pi_M}{dQ^2}=\lim_{Q\to+\infty}\left[2\int_{2Q}^{B}\frac{f(u)}{u}du\right]\cdot\left\{\lim_{Q\to+\infty}\frac{f(2Q)}{\int_{2Q}^{B}\frac{f(u)}{u}du}-2\right\}$$

$$=\lim_{Q\to+\infty}\left[2\int_{2Q}^{B}\frac{f(u)}{u}du\right]\cdot\left\{\lim_{Q\to+\infty}\frac{-\frac{df(2Q)}{dQ}}{\frac{d}{dQ}\int_{2Q}^{B}\frac{f(u)}{u}du}-2\right\} \qquad (25)$$

$$=\lim_{Q\to+\infty}\left[2\int_{2Q}^{B}\frac{f(u)}{u}du\right]\cdot\left\{-\lim_{Q\to+\infty}\frac{2Q\cdot f^{'}(2Q)}{f(2Q)}-2\right\}.$$

And, the third derivative $\frac{d^3\Pi_M}{dQ^3}$ equals

$$\frac{d^3\Pi_M}{dQ^3}=4f^{'}(2Q)+4\frac{f(2Q)}{Q}=\frac{2f(2Q)}{Q}\left[\frac{2Qf^{'}(2Q)}{f(2Q)}+2\right]. \qquad (26)$$

Now, *assume that* $\frac{2Qf^{'}(2Q)}{f(2Q)}+2$ *crosses the horizontal zero line (or changes sign) at most once*

(notice that this is a weaker condition than saying $\frac{2Qf^{'}(2Q)}{f(2Q)}$ is monotone).

Then, there are four cases to consider:

2a) $\frac{2Qf^{'}(2Q)}{f(2Q)}+2\geq 0$ for all Q;

2b) $\frac{2Qf^{'}(2Q)}{f(2Q)}+2\leq 0$ for all Q;

2c) $\lim_{Q\to 0^+}\frac{2Qf^{'}(2Q)}{f(2Q)}+2<0$ and $\lim_{Q\to+\infty}\frac{2Qf^{'}(2Q)}{f(2Q)}+2>0$;

2d) $\lim_{Q\to 0^+}\frac{2Qf^{'}(2Q)}{f(2Q)}+2>0$ and $\lim_{Q\to+\infty}\frac{2Qf^{'}(2Q)}{f(2Q)}+2<0$.

In cases 2a) and 2b), from (26) we have $\frac{d^3\Pi_M}{dQ^3}\geq 0$ and $\frac{d^3\Pi_M}{dQ^3}\leq 0$, respectively. Therefore, $\frac{d^2\Pi_M}{dQ^2}$ is monotone increasing or decreasing. Since, from (20) and (21), $\frac{d\Pi_M}{dQ}|_{Q\to 0^+}=1-c>0$ and $\frac{d\Pi_M}{dQ}|_{Q\to+\infty}\leq -c<0$, therefore, $\frac{d\Pi_M}{dQ}$ changes sign exactly once in the domain.

In case 2c), again from (26) we observe that $\frac{d^3\Pi_M}{dQ^3}$ changes sign from negative to positive once in the domain, so that $\frac{d^2\Pi_M}{dQ^2}$ decreases initially and then increases. Here we need to analyze three sub-cases:

for cases 1a) and 1b), that is, $f(0)=0$ and $f(0)>0$ with finite $f(0)$, we have $\lim_{Q\to 0^+} \frac{d^2\Pi_M}{dQ^2} < 0$.

Since, by (25), in case 2c) we have

$$\lim_{Q\to+\infty} \frac{d^2\Pi_M}{dQ^2} = \lim_{Q\to+\infty} \left[2\int_{2Q}^{B} \frac{f(u)}{u} du\right] \cdot \left\{-\lim_{Q\to+\infty} \frac{2Q\cdot f^{'}(2Q)}{f(2Q)} - 2\right\} \le 0,$$

we conclude that $\frac{d^2\Pi_M}{dQ^2} \le 0$ for all *Q*. Therefore, using (20) and (21) it follows that $\frac{d\Pi_M}{dQ}$ changes sign exactly once in the domain. For case 1c), we have

$$\lim_{Q\to 0^+} \frac{d^2\Pi_M}{dQ^2} = \lim_{Q\to 0^+} \left[2\int_{2Q}^{B} \frac{f(u)}{u} du\right] \cdot \left\{-\lim_{Q\to 0^+} \frac{2Q\cdot f^{'}(2Q)}{f(2Q)} - 2\right\} > 0, \text{ and}$$

$$\lim_{Q\to+\infty} \frac{d^2\Pi_M}{dQ^2} = \lim_{Q\to+\infty} \left[2\int_{2Q}^{B} \frac{f(u)}{u} du\right] \cdot \left\{-\lim_{Q\to+\infty} \frac{2Q\cdot f^{'}(2Q)}{f(2Q)} - 2\right\} \le 0,$$

which implies that $\frac{d^2\Pi_M}{dQ^2}$ decreases and crosses the horizontal zero line once (since $\frac{d^3\Pi_M}{dQ^3}$ changes sign once) and remains negative thereafter. Since, from (20) and (21), $\frac{d\Pi_M}{dQ}|_{Q\to 0^+} = 1-c > 0$ and $\frac{d\Pi_M}{dQ}|_{Q\to+\infty} \le -c < 0$, hence, $\frac{d\Pi_M}{dQ}$ changes sign exactly once.

In case 2d), similarly to case 2c) we can show that $\frac{d\Pi_M}{dQ}$ changes sign exactly once.

This completes the proof of the proposition. □

Derivation of p^* in Equation (4):

When $D(p) = e^{-p}$, with $\hat{u}$ as the realized value of *u*, and $z = Q/D(p)$, the retailer chooses his retail price *p* to maximize

$$\tilde{\Pi}_R = p\cdot D(p)\cdot \min\{z,\hat{u}\} - wQ = pe^{-p}\cdot \min\{z,\hat{u}\} - wQ.$$

If $z \le \hat{u}$, then $\frac{Q}{e^{-p}} \le \hat{u}$, that is, $p \le \ln(\hat{u}/Q) = \ln\hat{u} - \ln Q$. In this case,

$\tilde{\Pi}_R = pQ - wQ$, which is increasing in *p*; as *p* increases, $z = \frac{Q}{e^{-p}}$ increases as well, until we have

$z = \frac{Q}{e^{-p}} \geq \hat{u}$. This implies that the optimal stocking factor z^* cannot be less than $\hat{u}$. Hence, we only need to look for the optimal z^* with $z \geq \hat{u}$.

For $z \geq \hat{u}$, we have $\frac{Q}{e^{-p}} \geq \hat{u}$, that is, $p \geq \ln(\hat{u}/Q) = \ln \hat{u} - \ln Q$.

Then, the profit of the retailer becomes $\tilde{\Pi}_R = pe^{-p}\hat{u} - wQ$.

In this case, the first derivative of $\tilde{\Pi}_R$ equals $\frac{d\tilde{\Pi}_R}{dp} = \hat{u}e^{-p}(1-p)$, which attains its local maximum at p = 1. Since $p \geq \ln \hat{u} - \ln Q$ in this case, we have $p^* = \max\{1, \ln \hat{u} - \ln Q\}$. Letting $\ln \hat{u} - \ln Q = 1$, we get the break-even point $\hat{u} = eQ$, and hence, we obtain

$$p^* = \begin{cases} 1 & \text{if } \hat{u} \leq eQ, \\ \ln(\hat{u}/Q) & \text{if } \hat{u} \geq eQ. \end{cases}$$

Proof of Proposition 2.

(i) From (5) we get

$$\begin{aligned} \frac{d\tilde{\Pi}_R}{dQ} &= e^{-1} \cdot e \cdot eQ \cdot f(eQ) + \int_{eQ}^{B} \ln u \cdot f(u)du - Q \cdot e \cdot \ln(eQ) \cdot f(eQ) \\ &\quad - [\ln Q + 1]\bar{F}(eQ) + eQ \ln Q \cdot f(eQ) - w \\ &= \int_{eQ}^{B} \ln u \cdot f(u)du - [\ln Q + 1]\bar{F}(eQ) - w. \end{aligned}$$

Therefore,

$$\frac{d^2\tilde{\Pi}_R}{dQ^2} = -e \ln eQ \cdot f(eQ) - \frac{1}{Q}\bar{F}(eQ) + e\ln(eQ) \cdot f(eQ) = -\frac{1}{Q}\bar{F}(eQ) < 0,$$

that is, the retailer's expected profit function is strictly concave in Q.

(ii) Taking derivative of the manufacturer's expected profit function $\tilde{\Pi}_M(Q)$ given in (7), we obtain

$$\begin{aligned} \frac{d\tilde{\Pi}_M}{dQ} &= \left[\int_{eQ}^{B} \ln u \cdot f(u)du - \ln(eQ) \cdot \bar{F}(eQ) - c\right] - Q \cdot \frac{1}{Q}\bar{F}(eQ) \\ &= \int_{eQ}^{B} \ln u \cdot f(u)du - [2 + \ln Q]\bar{F}(eQ) - c, \end{aligned} \tag{27}$$

so that

$$\frac{d\tilde{\Pi}_M}{dQ}\Big|_{Q\to 0^+} = \int_{eQ}^{B} \ln u \cdot f(u)du - [2+\ln(eQ)]\bar{F}(eQ) - c \to +\infty, \tag{28}$$

and

$$\frac{d\tilde{\Pi}_M}{dQ}\Big|_{Q\to +\infty} = 0 - [2+\ln eQ]\bar{F}(eQ) - c < 0. \tag{29}$$

Moreover,

$$\begin{aligned}\frac{d^2\tilde{\Pi}_M}{dQ^2} &= -e\ln eQ \cdot f(eQ) - \frac{1}{Q}\bar{F}(eQ) + e[2+\ln Q]f(eQ) \\ &= ef(eQ) - \frac{1}{Q}\bar{F}(eQ) \\ &= \frac{1}{Q}\bar{F}(eQ)\left[\frac{eQf(eQ)}{\bar{F}(eQ)} - 1\right].\end{aligned} \tag{30}$$

Now, note that condition C in Proposition 2 corresponds to the following cases:

(i) $\frac{eQf(eQ)}{\bar{F}(eQ)} - 1 \geq 0$ for all Q;

(ii) $\frac{eQf(eQ)}{\bar{F}(eQ)} - 1 < 0$ for all Q;

(iii) $\lim_{Q\to 0^+} \frac{eQf(eQ)}{\bar{F}(eQ)} - 1 < 0$ and $\lim_{Q\to +\infty} \frac{eQf(eQ)}{\bar{F}(eQ)} - 1 > 0$;

(iv) $\lim_{Q\to 0^+} \frac{eQf(eQ)}{\bar{F}(eQ)} - 1 > 0$ and $\lim_{Q\to +\infty} \frac{eQf(eQ)}{\bar{F}(eQ)} - 1 < 0$.

According to (28) and (29), case (i) can be ruled out since by (30), $\tilde{\Pi}_M$ is convex and hence, its first derivative should be increasing.

For case (ii), $\frac{d^2\tilde{\Pi}_M}{dQ^2} < 0$ for all Q, so that $\tilde{\Pi}_M$ is strictly concave and hence, optimal Q is unique.

For case (iii), $\frac{d^2\tilde{\Pi}_M}{dQ^2}$ changes sign exactly once from negative to positive. Therefore, from (28) and (29), it follows that $\frac{d\tilde{\Pi}_M}{dQ}$ changes sign exactly once from positive to negative. Thus, optimal Q is unique.

For case (iv), the argument is similar to that of case (iii) above.

Therefore, we conclude that under condition C, $\tilde{\Pi}_M(Q)$ is unimodal in Q.

Now, from (27), the optimal Q^* is obtained by solving

$$c = \int_{eQ^*}^{B} \ln u . f(u) du - [2 + \ln Q^*] \bar{F}(eQ^*) ,$$

and therefore, from (6), the optimal wholesale price is given by

$$w^* = \int_{eQ}^{B} \ln u \cdot f(u) du - [\ln Q^* + 1] \bar{F}(eQ^*) = c + \bar{F}(eQ^*) . \quad \square$$

Derivation of p^* in Equation (9):

With $\hat{u}$ as the realized value of *u*, the retailer chooses his retail price *p* to maximize

$\tilde{\Pi}_R = p \cdot D(p) \cdot \min\{z, \hat{u}\} - wQ$,

where $z = Q / D(p)$. When $D(p) = p^{-q}$, $q > 1$,

if $z \le \hat{u}$, then $\frac{Q}{p^{-q}} \le \hat{u}$, that is, $p \le \left(\frac{\hat{u}}{Q} \right)^{1/q}$,

$\tilde{\Pi}_R = p \cdot D(p) \cdot \min\{z, \hat{u}\} = pQ - wQ$, which increases in *p* and attains the maximum at $p = \left(\frac{\hat{u}}{Q} \right)^{1/q}$;

if $z \ge \hat{u}$, then $p \ge \left(\frac{\hat{u}}{Q} \right)^{1/q}$, and

$\tilde{\Pi}_R = p \cdot D(p) \cdot \min\{z, \hat{u}\} = p^{1-q} \cdot \hat{u} - wQ$, which decreases in *p*, so that $\tilde{\Pi}_R$ still reaches the maximum at $p = \left(\frac{\hat{u}}{Q} \right)^{1/q}$. Hence, whatever the realization $\hat{u}$ is, $p^* = \left(\frac{\hat{u}}{Q} \right)^{1/q}$.

Proof of Proposition 3.

The retailer's expected profit function is given by

$$\tilde{\Pi}_R = \int_{A}^{B} \left(\frac{u}{Q} \right)^{1/q} \cdot \frac{Q}{u} \cdot u \cdot f(u) du - wQ = Q^{1-1/q} \int_{A}^{B} u^{1/q} \cdot f(u) du - wQ.$$

Therefore,

$$\frac{d\tilde{\Pi}_R}{dQ} = (1 - 1/q) \cdot Q^{-1/q} \int_{A}^{B} u^{1/q} \cdot f(u) du - w , \tag{31}$$

$$\frac{d^2\tilde{\Pi}_R}{dQ^2} = -\frac{q-1}{q^2} Q^{-1-1/q} \int_{A}^{B} u^{1/q} \cdot f(u) du < 0 ,$$

Hence, $\tilde{\Pi}_R$ is strictly concave in *Q*, and from (31), the optimal *w* is given by

$$w = (1 - 1/q) \cdot Q^{-1/q} \int_{A}^{B} u^{1/q} \cdot f(u) du . \tag{32}$$

Therefore, the expected profit function of the manufacturer is

$$\tilde{\Pi}_M = (w-c)Q = \left[(1-1/q)\cdot Q^{-1/q}\int_A^B u^{1/q}\cdot f(u)du - c\right]Q,$$

so that

$$\begin{aligned}\frac{d\tilde{\Pi}_M}{dQ} &= \left[(1-1/q)\cdot Q^{-1/q}\int_A^B u^{1/q}\cdot f(u)du - c\right] - \frac{q-1}{q^2}Q^{-1/q}\int_A^B u^{1/q}\cdot f(u)du \\ &= \frac{(q-1)^2}{q^2}Q^{-1/q}\int_A^B u^{1/q}\cdot f(u)du - c,\end{aligned} \tag{33}$$

and

$$\frac{d^2\tilde{\Pi}_M}{dQ^2} = -\frac{(q-1)^2}{q^3}Q^{-1-1/q}\int_A^B u^{1/q}\cdot f(u)du < 0,$$

which implies that $\tilde{\Pi}_M$ is strictly concave in *Q*. Therefore, Q^* is unique and from (33) it is obtained by solving: $c = \frac{(q-1)^2}{q^2}Q^{-1/q}\int_A^B u^{1/q}\cdot f(u)du$.

And, therefore, from (32) the optimal wholesale price is given as $w^* = \frac{qc}{q-1}$. □

Derivation of p^* in Equation (11):

When $D(p) = 1-p$, the retailer's profit with $\hat{u}$ as the realized value of u is given by

$$\Pi_R(Q) = p\cdot\min\{1-p+\hat{u}, Q\} - wQ.$$

If $Q < 1-p+\hat{u}$, then

$\Pi_R(Q) = p\cdot Q - wQ$, which is increasing in *p*, so that the retailer is better-off to increase *p* until $1-p+\hat{u} \le Q$, that is, $p \ge 1+\hat{u}-Q$; this means the optimal *p* does not exist if $Q < 1-p+\hat{u}$.

Therefore, for the optimal *p* we consider $Q \ge 1-p+\hat{u}$. Then,

$$\Pi_R(Q) = p\cdot(1-p+\hat{u}) - wQ,$$

which attains maximum at $p = \frac{1+\hat{u}}{2}$. Since $p \ge 1+\hat{u}-Q$ in this case, we have

$p^* = \max\left\{\frac{1+\hat{u}}{2}, 1+\hat{u}-Q\right\}$. Letting $1+\hat{u}-Q = \frac{1+\hat{u}}{2}$, we get the optimal *p* as

$$p^* = \begin{cases} \dfrac{1+\hat{u}}{2} & \text{if } \hat{u} \le 2Q-1 \\ 1+\hat{u}-Q & \text{if } \hat{u} \ge 2Q-1. \end{cases}$$

Proof of Proposition 4.

(i) From (12) we get

$$\frac{d\Pi_R}{dQ} = \int_{2Q-1}^{B} (1+u) f(u) du - 2Q \cdot \bar{F}(2Q-1) - w,$$

so that

$\dfrac{d^2\Pi_R}{dQ^2} = -2\bar{F}(2Q-1) < 0$. Hence, Π_R is strictly concave in Q.

(ii) Since the support of u is (A, B), where $A \ge 0$, we have, for $0 \le Q < 1/2$, $f(2Q-1) = 0$ and $\bar{F}(2Q-1) = 1$. Now, if u has an IFR distribution, then in (16) we have that $\dfrac{f(2Q-1)}{\bar{F}(2Q-1)}$ is non-decreasing in Q and so $\dfrac{2Qf(2Q-1)}{\bar{F}(2Q-1)}$ is strictly increasing in Q. Moreover,

$$\lim_{Q\to 0^+} \left[\frac{(2Q-1) f(2Q-1)}{\bar{F}(2Q-1)} + \frac{f(2Q-1)}{\bar{F}(2Q-1)} - 2 \right] = -2 < 0, \text{ and}$$

$$\lim_{Q\to +\infty} \left[\frac{(2Q-1) f(2Q-1)}{\bar{F}(2Q-1)} + \frac{f(2Q-1)}{\bar{F}(2Q-1)} - 2 \right] = +\infty - 2 > 0,$$

which means that $\dfrac{d^2\Pi_M}{dQ^2}$ changes sign only once, from negative to positive.

Now, we look at $\dfrac{d\Pi_M}{dQ}$ in (15). Since the expectation of demand X is positive, we have that

$$\frac{d\Pi_M}{dQ}\Big|_{Q=0} = \int_{-1}^{B} (1+u) f(u) du - 0 - c = 1 - c + E(u) > 1 - p + E(u) > 0, \text{ and}$$

$$\frac{d\Pi_M}{dQ}\Big|_{Q\to+\infty} = \lim_{Q\to+\infty} \left[\int_{2Q-1}^{B} (1+u) f(u) du - 4Q\bar{F}(2Q-1) - c \right] = -c < 0.$$

Thus, $\dfrac{d\Pi_M}{dQ}$ changes sign exactly once, from positive to negative. Therefore, we can conclude that if the demand distribution is IFR, then $\Pi_M(Q)$ is unimodal and there exists a unique Q^* which maximizes $\Pi_M(Q)$. □

Counter-examples of DFR/IGFR distributions for which $\Pi_M(Q)$ is not unimodal and $\frac{d^2\Pi_M}{dQ^2}$ is not monotone:

Consider the following gamma density function with shape parameter 2/3 and scale parameter 0.6,

$$f(u) = \frac{(0.6)^{2/3} \cdot u^{-1/3} \cdot e^{-(0.6)u}}{\Gamma(2/3)}.$$

Let the manufacturer's production cost *c* = 0.12.

In Figures 1a and 1b of the next page, we plot $\frac{d\Pi_M}{dQ}$ in the intervals [0.5, 0.54] and [0.495, 0.5] of *Q*, respectively. Also, in Figures 2a and 2b, we plot $\frac{d^2\Pi_M}{dQ^2}$ in the intervals [0.5, 0.54] and [0.495, 0.5] of *Q*, respectively. We have made separate plots for $Q \geq 0.5$ and $Q \leq 0.5$ because of the following reason: For the gamma distribution, if $0 \leq Q < 1/2$, then we have $f(2Q-1) = 0$ and $\bar{F}(2Q-1) = 1$. These make $\frac{d\Pi_M}{dQ}$ as piece-wise continuous with different expressions for $0 \leq Q \leq 1/2$ and $Q \geq 1/2$.

Similar behaviors are also observed for the Weibull density function $f(u) = (0.8) \cdot u^{-0.2} \cdot e^{-u^{0.8}}$, with shape parameter 0.8 and scale parameter 1, and with the manufacturer's production cost *c* = 0.138.

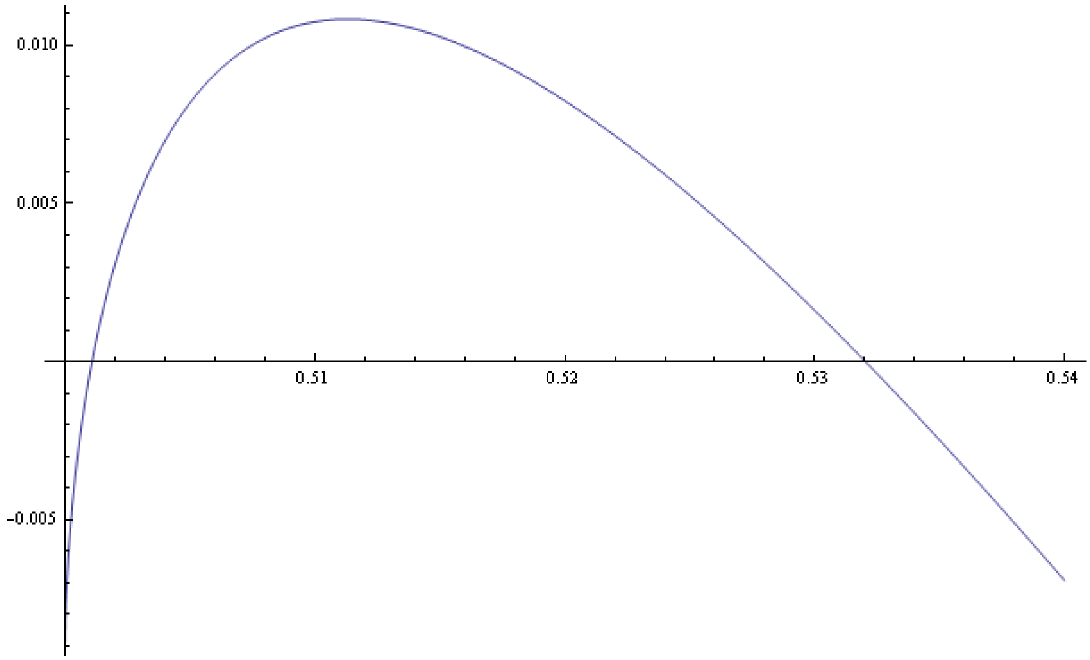

Figure 1a: Plot of $\dfrac{d\Pi_M}{dQ}$ in the interval [0.50, 0.54] of Q (for gamma distribution).

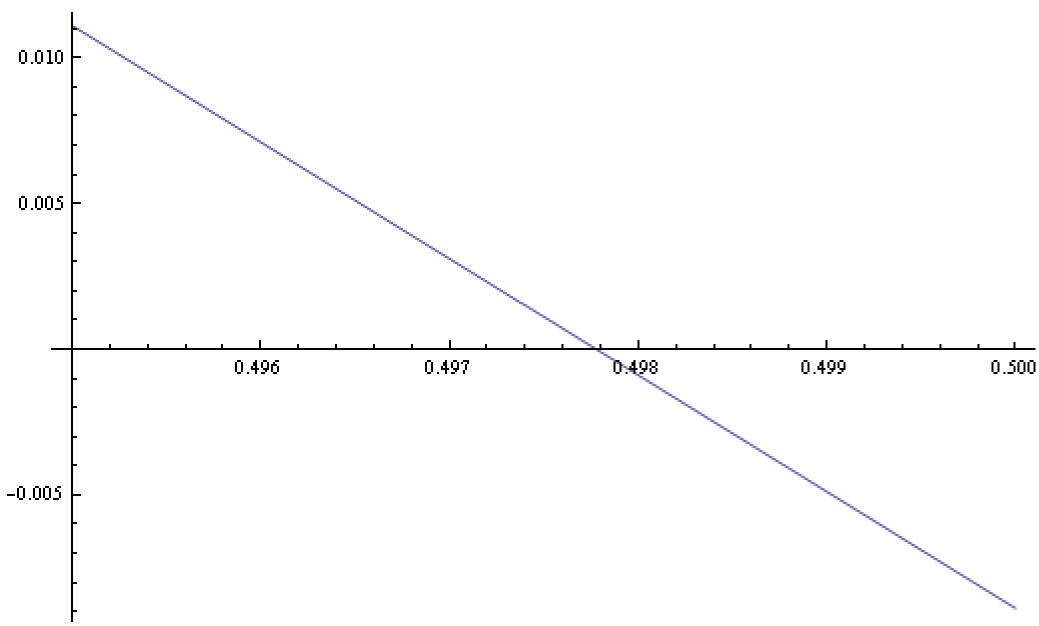

Figure 1b: Plot of $\dfrac{d\Pi_M}{dQ}$ in the interval [0.495, 0.50] of Q (for gamma distribution).

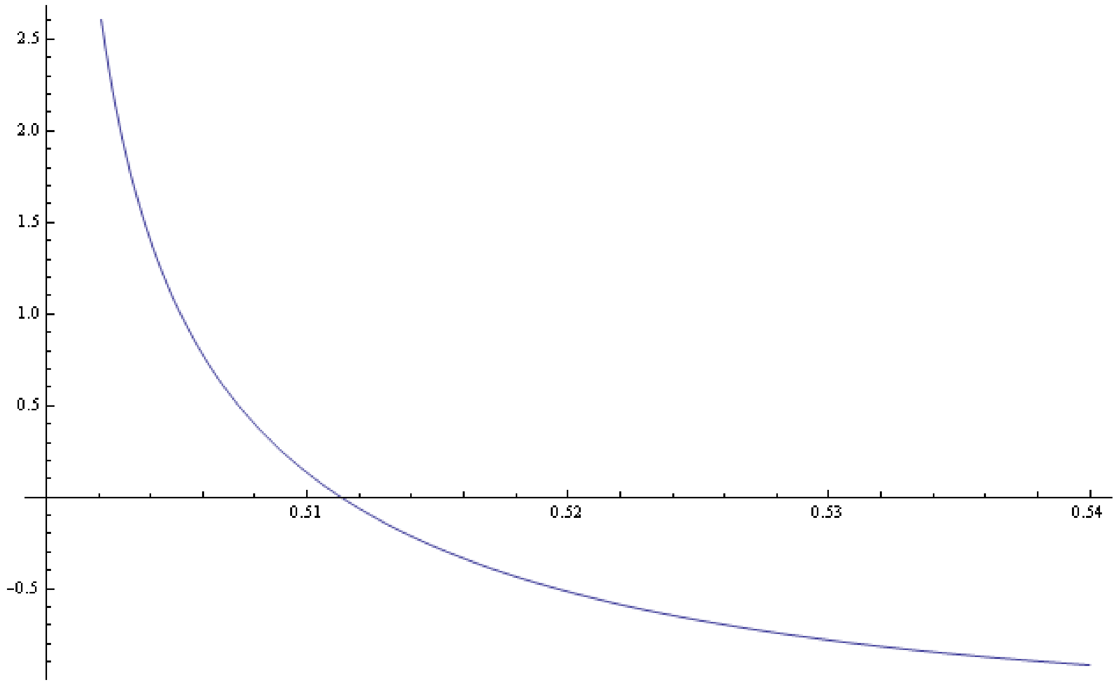

Figure 2a: Plot of $\frac{d^2\Pi_M}{dQ^2}$ in the interval [0.50, 0.54] of Q (for gamma distribution).

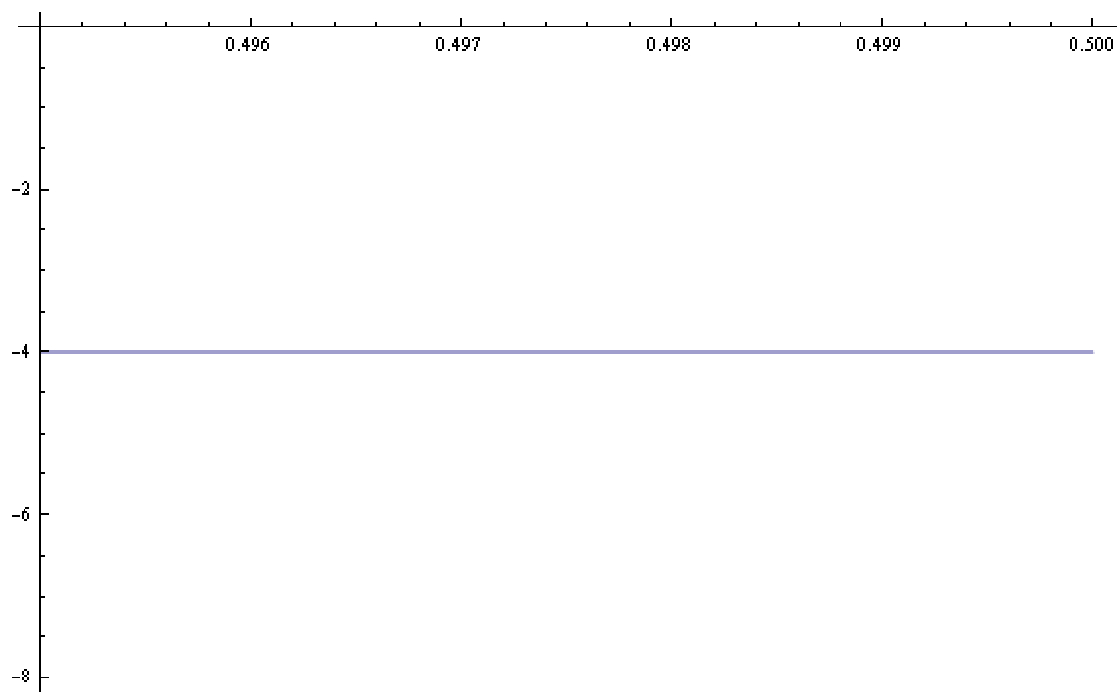

Figure 2b: Plot of $\frac{d^2\Pi_M}{dQ^2}$ in the interval [0.495, 0.5] of Q (for gamma distribution).

Explanation for the multi-modality of $\Pi_M(Q)$ and non-monotonicity of $\frac{d^2\Pi_M}{dQ^2}$ for DFR/IGFR (gamma & Weibull with shape parameter < 1) distributions:

From (15) and (16) we have

$$\frac{d\Pi_M}{dQ} = \int_{2Q-1}^{B} (1+u) f(u) du - 4Q\bar{F}(2Q-1) - c\,,$$

$$\frac{d^2\Pi_M}{dQ^2} = 4Qf(2Q-1) - 4\bar{F}(2Q-1).$$

Now, for gamma and Weibull distributions, for $0 \le Q < 1/2$, we have $f(2Q-1) = 0$ and $\bar{F}(2Q-1) = 1$.

Therefore, for $0 \le Q < 1/2$, $\frac{d\Pi_M}{dQ} = 1 + E(u) - 4Q - c$; and

$\frac{d\Pi_M}{dQ}|_{Q=0} = 1 - c + E(u) > 1 - p + E(u) > 0$, and $\frac{d\Pi_M}{dQ}|_{Q=1/2} = E(u) - 1 - c$.

If $c > E(u) - 1$, then $\frac{d\Pi_M}{dQ}|_{Q=1/2} = E(u) - 1 - c < 0$, which implies that $\frac{d\Pi_M}{dQ}$ changes sign once in [0, 1/2] (see Figure 1b). Moreover, for $0 \le Q < 1/2$, $\frac{d^2\Pi_M}{dQ^2} = -4$, as shown in Figure 2b.

Next, for $Q > 1/2$, $\frac{d^2\Pi_M}{dQ^2} = 4Qf(2Q-1) - 4\bar{F}(2Q-1)$, and

$$\lim_{Q\to 1/2^+} \frac{d^2\Pi_M}{dQ^2} = 2 \times \lim_{Q\to 1/2^+} f(2Q-1) - 4 = 2 \times \lim_{u\to 0^+} f(u) - 4\,.$$

For the gamma density with shape parameter $0 < \alpha < 1$, $\lim_{u\to 0^+} f(u) = \frac{\lambda^\alpha \cdot u^{\alpha-1} \cdot e^{-\lambda u}}{\Gamma(\alpha)} = +\infty$. Therefore, for Q slightly bigger than 1/2, $\frac{d^2\Pi_M}{dQ^2}$ will be a very large positive number (see Figure 2a). This implies that $\frac{d\Pi_M}{dQ}$ will increase dramatically and may become positive for $Q > 1/2$ (see Figure 1a). Then as Q increases, $\frac{d^2\Pi_M}{dQ^2}$ will decrease and may become negative, as can be seen from Figure 2a; consequently, $\frac{d\Pi_M}{dQ}$ will also decrease and may become negative (see Figure 1a).

Therefore, $\Pi_M(Q)$ can have multiple optima and $\frac{d^2\Pi_M}{dQ^2}$ can be non-monotone for some gamma distribution with shape parameter $0 < \alpha < 1$. The same is also true for some Weibull distribution with shape parameter $0 < \alpha < 1$.

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